Analysis of a Two-Branch Maximal Ratio and Selection Diversity System With Unequal SNRs and Correlated Inputs for a Rayleigh Fading Channel

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***Abstract*—An analytical expression for the probability density function of the signal-to-noise ratio (SNR) at the output of a two-branch maximal ratio and selection diversity system is given. The two branches are assumed to be Rayleigh fading, correlated, as well as of unequal average SNRs. Measurements of the cumulative distribution functions after selection and maximal ratio combining were made in Rayleigh fading channels and compared with the analytical results. Also presented are the exact analytical average probabilities of symbol error for coherent binary phase-shift keying and coherent quaternary phase-shift keying before and after two-branch maximal ratio combining for a slow and flat fading correlated Rayleigh channel.**

***Index Terms*—Antenna diversity, binary phase-shift keying, correlated channels, maximal ratio, quaternary phase-shift keying, Rayleigh, selection diversity.**

I. Introduction

DIVERSITY is an effective method for increasing the received signal-to-noise ratio (SNR) in a flat fading environment without increasing transmitter power. The signal envelope received by an antenna in a multiple reflective and nondispersive medium can often be modeled as Rayleigh fading. This type of fading significantly deteriorates the communications link. This paper presents an expression for the probability density function (pdf) of the SNR at the output of a two-branch selection and maximal ratio combining system for a correlated flat and slow Rayleigh fading environment with unequal average branch powers. Measurements made in Rayleigh channels were compared with the analytical expressions of the distribution of the SNR at the output of the selection and maximal ratio combiner and show very good agreement (cumulative distribution functions (CDF) curves were within 1 dB of each other). To the author's best knowledge, no other paper that has performed a similar theoretical analysis has compared its theory to actual measured data. Additionally, an exact expression for the average probability of symbol error before and after maximal ratio combining for a two-branch correlated Rayleigh flat and slow fading channel will be presented for coherent binary phase-shift keying (BPSK) and coherent quaternary phase-shift keying (QPSK) modulation. The treatment of correlated Rayleigh channels has been analyzed extensively in literature, the novelty in the theoretical analysis presented here is that closed form solutions are obtained for equations that were previously only available in integral form.

There has been significant theoretical research reported in the area of diversity systems and combining techniques for Rayleigh fading channels [5], [8]–[15]. Some analysis was performed for Nakagami channels of which Rayleigh is a special case [11]–[15]. Most papers consider diversity systems with uncorrelated branch signals [8]–[15]. Few have addressed the problems of correlated and unbalanced branches and their effects on diversity performance [12]–[15]; and among these only maximal ratio combining was considered. References [12]–[15] present equations for the performance of a correlated diversity system that use maximal ratio combining with differential phase-shift keying (DPSK) and noncoherent frequency shift keying (FSK) as modulation schemes. Noncoherent and differential modulation schemes have exponential bit-error rate (BER) performances in additive white Gaussian noise (AWGN) and are, therefore, more readily integrable over Nakagami or Rayleigh fading channels. In [15], an expression for the BER of coherent BPSK is presented for a maximal ratio combining diversity system for correlated Nakagami channels but its result is left in integral form. All results in [12]–[15] are expressed as a function of power correlation coefficient (or envelope squared correlation) because the joint pdf of the signals in the diversity system can be directly expressed in terms of this variable. The envelope correlation coefficient (or ρ), on the other hand, is the parameter that is commonly extracted from measurements and consequently the theory developed in this paper is based on this variable. An equation that relates the power correlation (ρ_P) to the envelope correlation (ρ) is given in this report. [16] and [17] have presented expressions for the pdf after maximal ratio combining for the Rayleigh channel once the eigenvalues are extracted from the correlations matrix. The pdf of dual selection diversity for correlated branches in a Rayleigh channel is presented in [18] and [19] but in both cases the result is left in integral form.

Measurements with different antennas or antenna configurations have shown that it is common for a two-branch antenna diversity system to have both received signals differ in average power from as low as 0 dB, for spatial diversity, to around 13 dB for polarization diversity [2], [3]. In a companion effort, the authors have made extensive measurements that produced similar

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results at 2.05 GHz [4]. If the diversity antennas are closely spaced, it is very likely that the fading in both branches is correlated. This paper addresses the gains achieved from either selection diversity or maximal ratio combining on a two-branch diversity system in the presence of power imbalance and correlation between branches. In subsequent sections, an expression that describes the dependence of the average symbol error probability on the power imbalance and correlation for a Rayleigh fading channel is given. The results presented in this paper are primarily directed toward antenna diversity systems but are not limited to these applications. The results can be used for other two-branch diversity mechanisms such as frequency or time diversity as long as they are Rayleigh distributed.

II. Analysis

The joint pdf of two correlated and unbalanced Rayleigh signals r_1 and r_2 can be found to be

$$\begin{aligned} f_{R_1R_2}(r_1,r_2) =& \frac{r_1 r_2}{\sigma_1^2\sigma_2^2(1-\rho_P)} e^{-\left(1/\left(2\sigma_1^2\sigma_2^2(1-\rho_P)\right)\right)\left[\sigma_2^2 r_1^2+\sigma_1^2 r_2^2\right]} \\ & \cdot I_0\left(\frac{r_1 r_2\sqrt{\rho_P}}{\sigma_1\sigma_2(1-\rho_P)}\right) \\ r_1 \geq & 0, r_2 \geq 0 \end{aligned} \tag{1}$$

$$\begin{aligned} f_{R_1}(r_1) &= \int_0^\infty f_{R_1R_2}(r_1,r_2)\,dr_2 = \frac{r_1}{\sigma_1^2} e^{-r_1^2/\left(2\sigma_1^2\right)} \\ f_{R_2}(r_2) &= \int_0^\infty f_{R_1R_2}(r_1,r_2)\,dr_1 = \frac{r_2}{\sigma_2^2} e^{-r_2^2/\left(2\sigma_2^2\right)}. \end{aligned} \tag{2}$$

The subscripts *1* and *2* on R, r and σ correspond to one of the two diversity branches (branch 1 and branch 2). Random variables R_1 and R_2 are Rayleigh distributed with average powers of $2\sigma_1^2$ and $2\sigma_2^2$, respectively, as can be seen from (2). $I_0(\bullet)$ is the modified Bessel function of order zero. Finally, ρ_P is the power envelope correlation and is defined as

$$\rho_P = \frac{E\left[\left(r_1^2-\overline{r_1^2}\right)\left(r_2^2-\overline{r_2^2}\right)\right]}{\sqrt{E\left[\left(r_1^2-\overline{r_1^2}\right)^2\right]E\left[\left(r_2^2-\overline{r_2^2}\right)^2\right]}} \tag{3}$$

were $E[\bullet]$ is the expected value operator. For equal branch powers ($2\sigma_1 = 2\sigma_2$), (1) reduces to an expression found in [7]. Measurement data is usually expressed in terms of the envelope correlation (ρ) instead of the power envelope correlation (ρ_P). The envelope correlation ρ is defined as and can be computed from the power correlation as follows:

$$\begin{aligned} \rho =& \frac{E\left[(r_1-\overline{r_1})(r_2-\overline{r_2})\right]}{\sqrt{E\left[(r_1-\overline{r_1})^2\right]E\left[(r_2-\overline{r_2})^2\right]}} \\ =& \frac{\left(1+\sqrt{\rho_P}\right)E\left(\frac{2\rho_P^{1/4}}{1+\sqrt{\rho_P}}\right)-\frac{\pi}{2}}{2-\frac{\pi}{2}} \end{aligned} \tag{4}$$

for the correlated Rayleigh channel where $E(\bullet)$ is the complete elliptical integral of the second kind (under the definition of the Jacobi elliptical integral with modulus k^2) and $E[\bullet]$ is the expected value operator. Subsequent expressions will be given as a function of power envelope correlation because of simplicity but the corresponding value of ρ can be computed from (4).

A. Two-Branch Selection Diversity

The results of selection diversity for a two-branch correlated Rayleigh fading channel are presented in this section. The branch with the largest SNR at any time is selected and connected to the receiver from a collection of possible diversity branches.

For a two-branch selection diversity system, the output power SNR, SNR_{P_S}, is given by

$$\mathrm{SNR}_{P_S} = \max\left(\frac{r_1^2}{N},\frac{r_2^2}{N}\right) = \frac{1}{N}\max\left(r_1^2,r_2^2\right) \tag{5}$$

were N is the noise power present in each of the two branches. For a Rayleigh channel, r_1 and r_2 are the envelopes of the signal received at both branches. At times it is better to analyze the problem using the voltage SNR (SNR_V), this is especially true when comparing theory to measured data. The SNR_V is defined as the square root of the SNR_P and for at two-branch selection diversity system is given by

$$\mathrm{SNR}_{V_S} = \sqrt{\mathrm{SNR}_{P_S}} = \frac{1}{\sqrt{N}}\max(r_1,r_2). \tag{6}$$

In literature, papers that quantify the performance of diversity systems through measurement apply the combining algorithm directly to the envelopes of the received signals. This is sufficient information when examining the gain of a diversity system over a single branch receiver when the system is known to have equal noise power in all branches. Mathematically, it represents the event when N is assumed to be unity and for a two-branch selection diversity system the voltage SNR after selection diversity is given by

$$s = \mathrm{SNR}_{V_S}\big|_{N=1} = \max(r_1,r_2). \tag{7}$$

Under the assumptions of $N = 1$, the voltage SNR at each branch, SNR_{V_1} and SNR_{V_2}, has the same distributed as the envelope (r_1 or r_2) of that particular branch

$$\mathrm{SNR}_{V_1}\big|_{N=1} = r_1,\ \mathrm{SNR}_{V_2}\big|_{N=1} = r_2. \tag{8}$$

The pdf of s in (7), $f_S(s)$ as a function of ρ_P and average branch powers for a Rayleigh channel can be computed probabilistically using (7) and (1). The resulting pdf for $f_S(s)$ for a two-branch correlated Rayleigh channel with unequal branch power for ρ less than one is given by (9) shown at the bottom of the next page. The value of η determines the choice of $J(x,y)$ to complete the equation for $f_S(s)$. $I_k(\xi)$ is the modified Bessel function of order k. For the special case when the envelopes of both branches are perfectly correlated, or $\rho = \rho_P = 1$, the pdf $f_S(s)$ takes on the form

$$\begin{aligned} f_S(s) =& \frac{s}{\max(\sigma_1^2,\sigma_2^2)} e^{-s^2/\left(2\max(\sigma_1^2,\sigma_2^2)\right)} \\ s \geq & 0, \rho = \rho_P = 1. \end{aligned} \tag{10}$$

This representation is equivalent to connecting the branch with the largest average power directly to the receiver.

The distribution of the output SNRs in (5) and (6) as a function of N can be computed by applying a transformation of variables to the distributions in (9) and (10) that can be found in most introductory books on random processes [6].

B. Two-Branch Maximal Ratio Combining

When the envelopes received by a two-branch diversity system are given by r_1 and r_2 and both branches have a noise power of N, the voltage SNR after maximal ratio combining can be found to be

$$\mathrm{SNR}_{V_M} = \sqrt{\mathrm{SNR}_{P_M}} = \frac{\sqrt{r_1^2 + r_2^2}}{\sqrt{N}}. \tag{11}$$

As stated earlier, to produce a theoretical result that is consistent with what is commonly done with measured diversity data, the theoretical distribution of the SNR_{V_M} for a noise power of unity is required. In measurement, the recorded envelope data is combined in time for a two-branch maximal ratio combiner as follows:

$$m = SNR_{V_M}|_{N=1} = \sqrt{r_1^2 + r_2^2}. \tag{12}$$

As with selection, when N is unity, the distribution of the input voltage SNR is the distribution of branch envelopes as shown in (8). Most of the measurement work is done to quantify the improvement of a diversity system over a single branch. This gain can be shown to be independent of the value N so the actual value of the noise power is irrelevant for this type of analysis.

The pdf of m in (12), can be found using the distribution in (1) to arrive at the pdf of the SNR_V at the output of the maximal ratio combiner or $f_M(m)$. Three equations are needed to describe the pdf $f_M(m)$ over all possible values of ρ, ρ_P, σ_1, and σ_2.

Equation (13) found at the bottom of the page, is valid for partially correlated signals ($\rho < 0$) where branch 2 is the branch with the largest average power of both channels ($\sigma_1 \leq \sigma_2$). Equation (13) cannot be used, however, when both branches have equal average power ($\sigma_1 = \sigma_2$) and at the same time are uncorrelated ($\rho = \rho_P = 0$). Under the special circumstances when the average branch powers are equal and uncorrelated ($\sigma_1 = \sigma_2 = \sigma$ and $\rho_P = \rho = 0$), the distribution of $f_M(m)$ results in the following:

$$\begin{aligned} f_M(m) =& \frac{m^3}{2\sigma^4} e^{-(m^2/(2\sigma^2))} \\ m \geq& 0 \\ \text{valid for } \sigma_1 =& \sigma_2 = \sigma \text{ and } \rho = \rho_P = 0. \end{aligned} \tag{14}$$

Finally, when the branches are perfectly correlated ($\rho = \rho_P = 1$) the pdf of $f_M(m)$ is described by

$$\begin{aligned} f_M(m) =& \frac{m}{(\sigma_1^2 + \sigma_2^2)} e^{-(m^2/(2(\sigma_1^2+\sigma_2^2)))} \quad m \geq 0 \text{ and} \\ \rho =& \rho_P = 1. \end{aligned} \tag{15}$$

For completeness, the cumulative distribution function (cdf) of $f_M(m)$ can be derived by integrating the pdfs in (13), (14), and (15) over the limits from 0 to m. The resulting cdf ($F_M(m)$) for the three cases listed above are shown in (16)–(18) at the bottom of the next page. The envelope correlation ρ can be computed from ρ_P by using (4). The distributions of the SNR_{V_M} and SNR_{P_M}, which are a function of N, can be computed from (13), (14), and (15) through a simple transformation as found in [6].

1) Two-Branch Diversity With Maximal Ratio Combining and BPSK Modulation: A flat fading channel occurs when all frequency components of the signal of interest are equally affected in amplitude by the channel and only a linear phase shift over frequency is introduced. A flat fading channel translates into an environment in which the multipath components arrive at the receiver at almost the same time.

$$\begin{aligned} f_S(s) =& \frac{s}{\sigma_1^2} e^{-s^2/(2\sigma_1^2)} \left[1 - J_1(x_1, y_1)\right] + \frac{s}{\sigma_2^2} e^{-s^2/2\sigma_2^2} \left[1 - J_2(x_2, y_2)\right] \; s \geq 0, \rho < 1 \\ \text{if } \eta_{1,2} =& 1, x_{1,2} = y_{1,2} \quad J_{1,2}(x_{1,2}, x_{1,2}) = \frac{1}{2}\left[1 + e^{-2x_{1,2}} I_0(2x_{1,2})\right] \\ \text{if } \eta_{1,2} <& 1 \quad J_{1,2}(x_{1,2}, y_{1,2}) = e^{-(x_{1,2}+y_{1,2})} \sum_{k=0}^{\infty} \eta_{1,2}^k I_k(\xi) \\ \text{if } \eta_{1,2} >& 1 \quad J_{1,2}(x_{1,2}, y_{1,2}) = 1 - e^{-(x_{1,2}+y_{1,2})} \sum_{k=1}^{\infty} \eta_{1,2}^{-k} I_k(\xi) \\ \eta_{1,2} =& \frac{\sigma_1 \sigma_2 \sqrt{\rho_P}}{\sigma_{1,2}^2}, x_{1,2} = \frac{s^2 \sigma_{1,2}^2}{2\sigma_1^2 \sigma_2^2 (1-\rho_P)}, y_{1,2} = \frac{s^2 \rho_P}{2\sigma_{1,2}^2 (1-\rho_P)}, \xi = \frac{s^2 \sqrt{\rho_P}}{\sigma_1 \sigma_2 (1-\rho_P)} \end{aligned} \tag{9}$$

$$\begin{aligned} f_M(m) =& \frac{m e^{-\frac{m^2(\sigma_1^2+\sigma_2^2)}{4\sigma_1^2\sigma_2^2(1-\rho_P)}}}{\sqrt{(\sigma_2^2 - \sigma_1^2)^2 + 4\sigma_1^2\sigma_2^2\rho_P}} \left[e^{\frac{m^2 \sqrt{(\sigma_2^2-\sigma_1^2)^2 + 4\sigma_1^2\sigma_2^2\rho_P}}{4\sigma_1^2\sigma_2^2(1-\rho_P)}} - e^{-\frac{m^2 \sqrt{(\sigma_2^2-\sigma_1^2)^2 + 4\sigma_1^2\sigma_2^2\rho_P}}{4\sigma_1^2\sigma_2^2(1-\rho_P)}} \right] \\ m \geq& 0; \rho_P, \rho < 1 \\ \text{for } \sigma_2 \geq& \sigma_1 \text{ except when both, } \sigma_1 = \sigma_2 \text{ and } \rho = \rho_P = 0 \end{aligned} \tag{13}$$

The average probability of symbol error for a slow and flat channel P^e can be found by averaging P^e_{AWGN} (the probability of symbol error in a AWGN channel) over the distribution of $f_Y(y)$ (the pdf of E_s/N_0) as follows:

$$P^e = \int_0^\infty P^e_{\text{AWGN}}(y) f_Y(Y)\,dy. \tag{19}$$

The distribution of $f_Y(y)$, for a flat and slow fading channel is proportional to the SNR_P since both represent power SNRs. The average probability of coherent BPSK in a Rayleigh fading channel can be found using (19) to give

$$P^e_{\text{Rayleigh,BPSK}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2\sigma^2}{1+2\sigma^2}}, \qquad \frac{\overline{E}_b}{N_0} = 2\sigma^2. \tag{20}$$

This result has also been derived in [5] where the average $\overline{E}_b/N_0$ is given by $2\sigma^2$. This procedure can be repeated for the P^e after maximal ratio combining by first transforming the pdfs at the output of the maximal ratio combiner given by (13), (14), and (15) into their SNR_P counterparts. The average symbol error rate of coherent BPSK at the output of a two-branch maximal ratio combiner in a Rayleigh channel with correlated and unequal branch powers is given by (21)–(23) shown at the bottom of the page.

Equations (21)–(23) are the equations for the average symbol error rate for coherent BPSK after maximal ratio combining. The $\overline{(E_b/N_0)}_{\text{Max}}$ is the average E_b/N_0 at the output of the diversity configuration. As before, (21) is valid for all envelope correlations less than unity except when both branches are of equal power and at the same time uncorrelated. Equation (22) applies when both branches have the same average power and are uncorrelated. Finally, (23) is valid for the special case when both r_1 and r_2 are perfectly correlated. It can be seen from (21),

$$\begin{aligned} F_M(m) = 1 - &\frac{e^{-\frac{m^2(\sigma_1^2+\sigma_2^2)}{4\sigma_1^2\sigma_2^2(1-\rho_P)}}}{2\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}} \\ &\cdot \left[\begin{array}{l} e^{\frac{m^2\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}{4\sigma_1^2\sigma_2^2(1-\rho_P)}}\left(\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}+(\sigma_1^2+\sigma_2^2)\right) \\ +e^{-\frac{m^2\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}{4\sigma_1^2\sigma_2^2(1-\rho_P)}}\left(\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}-(\sigma_1^2+\sigma_2^2)\right)\end{array}\right] \\ &m \geq 0; \rho, \rho_P < 1 \\ &\text{for } \sigma_2 \geq \sigma_1 \text{ except when both, } \sigma_2 = \sigma_1 \text{ and } \rho = \rho_P = 0 \end{aligned} \tag{16}$$

$$F_M(m) = 1 - \left(1 + \frac{m^2}{2\sigma^2}\right)e^{-(m^2/(2\sigma^2))} \quad m \geq 0, \text{ valid for } \sigma_2 = \sigma_1 = \sigma \text{ and } \rho = \rho_P = 0 \tag{17}$$

$$F_M(m) = 1 - e^{-(m^2/(2(\sigma_1^2+\sigma_2^2)))} \; m \geq 0 \text{ and } \rho = \rho_P = 1 \tag{18}$$

$$\begin{aligned} P^e_{\text{Max,BPSK}} = \frac{1}{2} - &\frac{\left((\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)}{4\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}\sqrt{\frac{(\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}{1+(\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}} \\ + &\frac{\left((\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)}{4\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}\sqrt{\frac{(\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}{1+(\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}} \\ \left(\frac{\overline{E}_b}{N_0}\right)_{\text{Max}} = &2(\sigma_1^2+\sigma_2^2) \text{ valid for } \rho, \rho_P < 1, \\ &\text{for } \sigma_2 \geq \sigma_1 \text{ except when both, } \sigma_2 = \sigma_1 \text{ and } \rho = \rho_P = 0 \end{aligned} \tag{21}$$

$$P^e_{\text{Max,BPSK}} = \frac{1}{2} - \frac{\sqrt{2}}{4}\left[\frac{(3\sigma+4\sigma^3)}{(1+2\sigma^2)^{3/2}}\right], \quad \left(\frac{\overline{E}_b}{N_0}\right)_{\text{Max}} = 2(\sigma_1^2+\sigma_2^2) = 4\sigma^2 \tag{22}$$

valid for $\sigma_2 = \sigma_1 = \sigma$ and $\rho = \rho_P = 0$

$$P^e_{\text{Max,BPSK}} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2(\sigma_1^2+\sigma_2^2)}{1+2(\sigma_1^2+\sigma_2^2)}}, \quad \left(\frac{\overline{E}_b}{N_0}\right)_{\text{Max}} = 2(\sigma_1^2+\sigma_2^2)$$

valid for $\rho = \rho_P = 1$ (23)

(22), and (23) that the average E_b/N_0 after maximal ratio combining is given by

$$\left(\frac{\overline{E}_b}{N_0}\right)_{\text{Max}} = \left(\frac{\overline{E}_b}{N_0}\right)_{\text{Branch1}} + \left(\frac{\overline{E}_b}{N_0}\right)_{\text{Branch2}}. \tag{24}$$

The following section addresses the performance of coherent QPSK for a two-branch diversity system using maximal ratio combining.

2) Two-Branch Diversity With Maximal Ratio Combining and QPSK Modulation: The probability of symbol error for coherent QPSK in an AWGN channel is given in [5] as

$$P^e_{\text{AWGN,QPSK}} = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) - \left[Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right]^2. \tag{25}$$

As was done for coherent BPSK, (19) can be used to compute average symbol error rate for coherent QPSK in a Rayleigh fading channel. Using (25) in (19) produces the following expression for the average probability of symbol error of coherent QPSK for a Rayleigh channel

$$P^e_{\text{Rayleigh,QPSK}} = \frac{3}{4} - \sqrt{\frac{\sigma^2}{(1+\sigma^2)}} \cdot \left[\frac{1}{2} + \frac{1}{\pi}\arctan\sqrt{\frac{\sigma^2}{(1+\sigma^2)}}\right] \tag{26}$$

with an average E_s/N_0 of

$$\frac{\overline{E}_s}{N_0} = 2\sigma^2. \tag{27}$$

The above and subsequent symbol error rate equations for coherent QPSK presented here were derived using the expression of symbol error rate given in (25) and not the union bound approximation. Similarly, the average symbol error probability of coherent QPSK after two-branch maximal ratio combining can be computed using the densities given in (13)—(15). The following three equations describe the average symbol error probability of coherent QPSK after two-branch maximal ratio combing for a correlated Rayleigh fading channel with unbalanced branches as shown in (28)–(30) at the bottom of the page. Equations (28)–(30) yield the average symbol error probability after maximal ratio combining for coherent QPSK and are valid over the intervals listed with the equations.

C. Measurement Results and Comparison With Theory

Measurements were taken in an indoor non line of sight environment with a four-channel receiver. The receiver consisted of four vertically polarized dipole antennas separated by spacings of 0.2λ at a frequency of 2.05 GHz. Fig. 1 shows the receiver system with the four receive antennas and the vertically transmitting dipole antenna that was broadcasting a constant tone. Measurements were made over a distance of approximately 50 m at walking speeds. The long term variations of the average received signal level were removed during post processing and all received envelopes were normalized to the mean of the strongest branch. The recorded envelope measurements at the four ports were very much Rayleigh distributed as can be seen from Figs. 2 and 3. Selection and maximal ratio combining were performed on two branches at a time off-line and compared with the theory presented in this paper. The time average power of the individual branches after normalization was computed and this value was used as the average power of the Rayleigh distributed signal for that branch (2). The computed time envelope correlation between signals r_1 and r_2 was used as the probabilistic envelope corre-

$$\begin{aligned}
P^e_{\text{Max,QPSK}} =& \frac{3}{4} - \frac{1}{4\sqrt{(\sigma_2^2-\sigma_1^2)^2 + 4\sigma_1^2\sigma_2^2\rho_P}} \\
&\cdot \left[\frac{\left((\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)^{3/2}}{\sqrt{2+(\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}} \left(1+\frac{2}{\pi}\arctan\sqrt{\frac{\left((\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)}{2+(\sigma_1^2+\sigma_2^2)+\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}}\right) \right. \\
&\left. - \frac{\left((\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)^{3/2}}{\sqrt{2+(\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}} \left(1+\frac{2}{\pi}\arctan\sqrt{\frac{\left((\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}\right)}{2+(\sigma_1^2+\sigma_2^2)-\sqrt{(\sigma_2^2-\sigma_1^2)^2+4\sigma_1^2\sigma_2^2\rho_P}}}\right) \right] \\
\left(\frac{\overline{E}_s}{N_0}\right)_{\text{Max}} =& 2\left(\sigma_1^2+\sigma_2^2\right) \text{ valid for } \rho, \rho_P < 1 \\
&\text{for } \sigma_2 \geq \sigma_1 \text{ except when both, } \sigma_2=\sigma_1 \text{ and } \rho=\rho_P=0
\end{aligned} \tag{28}$$

$$\begin{aligned}
P^e_{\text{Max,QPSK}} =& \frac{3}{4} - \frac{3\sigma+2\sigma^3}{4(1+\sigma^2)^{3/2}}\left[1+\frac{2}{\pi}\arctan\sqrt{\frac{\sigma^2}{1+\sigma^2}}\right] - \frac{\sigma^2}{2\pi(1+2\sigma^2)(1+\sigma^2)} \\
\left(\frac{\overline{E}_s}{N_0}\right)_{\text{Max}} =& 2\left(\sigma_1^2+\sigma_2^2\right) = 4\sigma^2 \text{ valid for } \sigma_2=\sigma_1=\sigma \text{ and } \rho=\rho_P=0
\end{aligned} \tag{29}$$

$$\begin{aligned}
P^e_{\text{Max,QSPK}} =& \frac{3}{4} - \sqrt{\frac{\sigma_1^2+\sigma_2^2}{(1+\sigma_1^2+\sigma_2^2)}}\left[\frac{1}{2}+\frac{1}{\pi}\arctan\sqrt{\frac{\sigma_1^2+\sigma_2^2}{(1+\sigma_1^2+\sigma_2^2)}}\right] \\
\left(\frac{\overline{E}_s}{N_0}\right)_{\text{Max}} =& 2\left(\sigma_1^2+\sigma_2^2\right) \text{ valid for } \rho=\rho_P=1
\end{aligned} \tag{30}$$

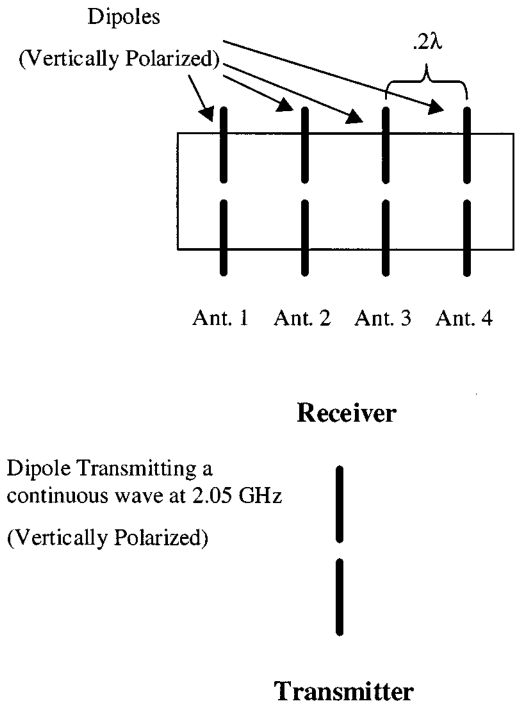

Fig. 1. Diagram of the transmitter and receiver system.

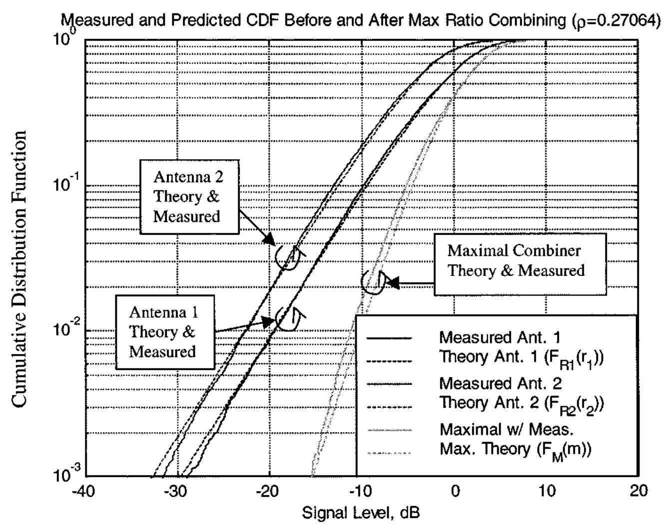

Fig. 2. Measured and theoretical cdf for antenna 1 and 2 and the maximal ratio combining result.

lation ρ (4). The predicted cdf for selection ($F_S(s)$) and maximal ratio combining ($F_M(m)$) was computed using the average powers of both branches and the envelope correlation ρ. The resulting $F_S(s)$ and $F_M(m)$ were compared with the distributions at the output of the selection and maximal ratio combiner using combining in the time domain on the measured data. The time average power of the signals at antenna 1–4 were computed to be

$$\begin{aligned}\overline{r}_1^2 &=1.08 = E\left[r_1^2\right] = 2\sigma_1^2\\ \overline{r}_2^2 &=0.53 = E\left[r_2^2\right] = 2\sigma_2^2\\ \overline{r}_3^2 &=0.78 = E\left[r_3^2\right] = 2\sigma_3^2\\ \overline{r}_4^2 &=11.12 = E\left[r_4^2\right] = 2\sigma_4^2\end{aligned}$$

and these values were used for the variances of the Rayleigh pdfs. Figs. 2 and 4 show the results when the envelope at antenna 1 and

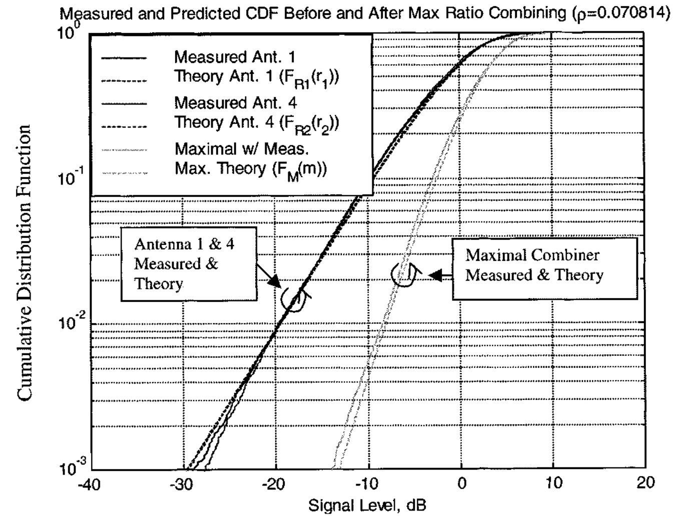

Fig. 3. Measured and theoretical cdf for antenna 1 and 4 and the maximal ratio combining result.

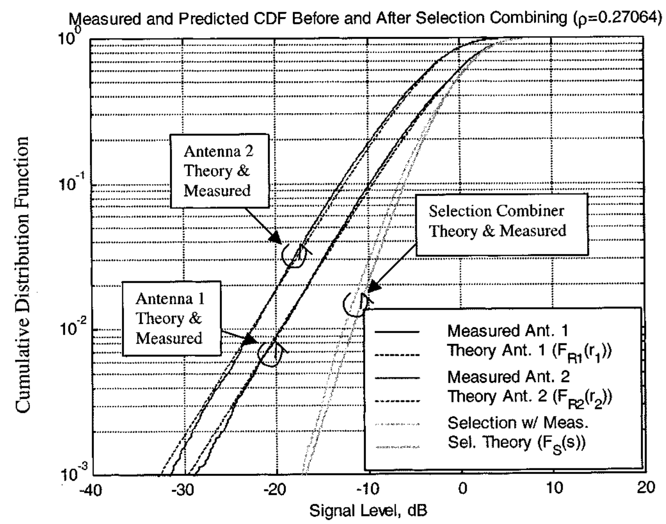

Fig. 4. Measured and theoretical cdf for antenna 1 and 2 and the selection combining result.

antenna 2 are used together in a two-branch diversity system using selection and maximal ratio combining. The time envelope correlation between antenna 1 and 2 was computed to be 0.271 (which was used for ρ) and the predicted CDF after combining ($F_S(s)$ and $F_M(m)$) closely follows the distribution achieved through time combining on the measured data. Figs. 3 and 5 are plots of the measured and predicted cdfs before and after combining using antennas 1 and 4 which are the elements that are furthest away in distance. The envelope correlation between antennas 1 and 4 was calculated to be 0.071 from measured data and as before the predicted CDF after combining matches the distribution using time combining fairly well. The predicted results shown in Figs. 2–5 closely match the results achieved through measurements and the CDFs consistently fall within 1 dB of each other. The difference in power measured by the two center elements compared with the outer elements is probably due to antenna coupling. The center elements are both surrounded by antennas on both sides as opposed to just one for the outer elements. More information on antenna patterns affected by coupling can be found in [4].

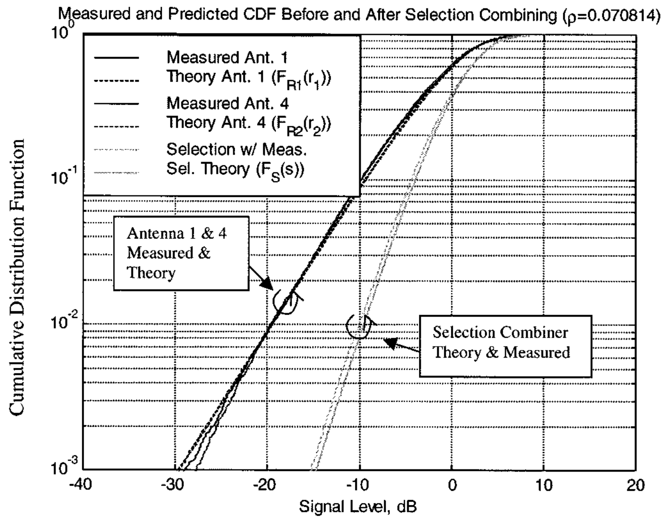

Fig. 5. Measured and theoretical cdf for antenna 1 and 4 and the selection combining result.

The results presented in this section are characteristic of the results obtained during the measurement campaign. For all measurements, the differences between the theoretical and measured CDFs did not exceed 1 dB at any point along the curve. The small differences between what was measured and what was expected is probably due to the small variations the measurements results had from the ideal Rayleigh distribution. The deviation from the ideal Rayleigh distribution is worse at the low values of the CDF curves; the effects of these departures are observed further up on the CDF of the combined signal. The estimates of r_1 and r_2 are usually worse at the lower levels where the effect of the noise becomes more noticeable.

III. Conclusion

The pdfs of the SNR after selection and maximal ratio combining were presented for a two-branch diversity system. The inputs to the system are two correlated Rayleigh distributed signals with unequal branch powers. Additionally, the symbol error rates for coherent QPSK and BPSK were given at the input and at the output of a two-branch maximal ratio combining diversity system for this Rayleigh environment under the assumptions of flat and slow fading. All these expressions are a function of ρ_P, σ_1, and σ_2. $2\sigma_1$ and $2\sigma_2$ determine the average power of channel *1* and *2*, respectively. ρ_P describes the power correlation between the signals that have envelopes r_1 and r_2. All the equations presented in this paper have been verified through numerical integration to ensure that the results derived theoretically are accurate.

One of the assumptions made in this paper is that the ensemble operations over probability distributions agree with time averages applied to measured signals. The measurements shown in Section II-C show that there is a very good agreement between what is predicted using probability distributions and distributions achieved from measured data through time combining. The measurements show an agreement within 1 dB between the cdf predicted by theory and the distributions achieved from measured data. The performance of a two-branch selection and maximal ratio combiner in a Rayleigh channel can be predicted very accurately through the theoretical results presented in this paper. Additionally, a comparison of diversity gain at the 10% level (as a function of ρ and average power imbalance) for the selection and maximal ratio combiner in the Rayleigh channel shows a very good agreement with results reported by Turkmani *et al.* [2] for an average radio channel at 1800 MHz.

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References

[1] R. H. Clarke, "A statistical theory of mobile radio reception," *Bell Syst. Tech. J.*, vol. 47, pp. 957–1000, July 1968.
[2] A. M. D. Turkmani, A. A. Arowogolu, P. A. Jefford, and C. J. Kellent, "An experimental evaluation of performance of two-branch space and polarization diversity schemes at 1800 MHz," *IEEE Trans. Veh. Technol.*, vol. 44, pp. 318–326, May 1995.
[3] R. G. Vaughan, "Polarization diversity in mobile communications," *IEEE Trans.Veh. Technol.*, vol. 39, pp. 177–186, Aug. 1990.
[4] C. B. Dietrich Jr., K. Dietze, J. R. Nealy, and W. L. Stutzman, "Spatial, polarization and pattern diversity for wireless handheld terminals," *IEEE Trans. Antennas Propagat.*, vol. 49, pp. 1271–1281, Sept. 2001.
[5] J. G. Proakis, *Digital Communications*. New York: McGraw-Hill, 1995.
[6] H. Stark and J. W. Woods, *Probability, Random Processes and Estimation Theory for Engineers*. Englewood Cliffs, NJ: Perentice-Hall, 1994.
[7] W. C. Jakes, *Microwave Mobile Communications*, W. C. Jakes, Ed. New York: Wiley, 1974.
[8] T. Eng, N. Kong, and L. B. Milstein, "Comparison of diversity combining techniques for Rayleigh-fading channels," *IEEE Trans. Commun.*, vol. 44, pp. 1117–1129, Sept. 1996.
[9] N. Kong and L. B. Milstein, "Combined average SNR of a generalized diversity selection combining scheme," in *Proc. IEEE ICC98*, Atlanta, GA, June 1998, pp. 1556–1560.
[10] J. Lu, T. T. Tjhung, and C. C. Chai, "Error probability of L-branch diversity reception of MQAM in Rayleigh fading," *IEEE Trans. Commun.*, vol. 46, pp. 179–181, Feb. 1998.
[11] V. Aalo and S. Pattaramalai, "Average error rate for coherent MPSK signals in Nakagami fading channels," *IEE Electron. Lett.*, vol. 32, pp. 1538–1539, Aug. 1996.
[12] E. K. Al-Hussaini and A. A. M. Al-Bassiouni, "Performance of MRC diversity systems for the detection of signals with Nakagami fading," *IEEE Trans. Commun.*, vol. COM-33, pp. 1315–1319, Dec. 1985.
[13] F. Patenaude, J. H. Lodge, and J.-Y. Chouinard, "Noncoherent diversity reception over Nakagami-fading channels," *IEEE Trans. Commun.*, vol. 46, pp. 985–991, Aug. 1998.
[14] Q. T. Zhang, "Exact analysis of postdetection combining for DPSK and NFSK system over arbitrarily correlated Nakagami channels," *IEEE Trans. Commun.*, vol. 46, pp. 1459–1467, Nov. 1998.
[15] P. Lombardo, G. Fedele, and M. M. Rao, "MRC performance for binary signals in Nakagami fading with general branch correlation," *IEEE Trans. Commun.*, vol. 47, pp. 44–52, Jan. 1999.
[16] O. Norklit and R. G. Vaughan, "Method to determine effective number of diversity branches," in *Proc. Global Communications Conf. 1998*, pp. 138–141.
[17] M. Z. Win and J. H. Winters, "On maximal ratio combining in correlated Nakagami channels with unequal fading parameters and SNRs among branches: An analytic framework," in *Proc. Wireless Communications and Networking Conf.*, vol. 3, 1999, pp. 1058–1064.
[18] F. Adachi, K. Ohno, and M. Ikura, "Postdetection selection diversity reception with correlated, unequal average power Rayleigh fading signals for pi/4-shift QDPSK mobile radio," *IEEE Trans. Veh. Technol.*, vol. 41, pp. 199–210, May 1992.
[19] M. K. Simon and M.-S. Alouini, "A unified performance analysis of digital communications with dual selective combining diversity over correlated Rayleigh and Nakagami-m fading channels," *IEEE Trans. Commun.*, vol. 47, pp. 33–43, Jan. 1999.

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